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Heavy quarks on the lattice: status and perspectives

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Summary. — The lattice method for the calculation of weak decay amplitudes of heavy quark systems is introduced. Results for leptonic and semi-leptonic decays of heavy mesons and $B-\overline{B}$ mixing are reviewed.

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Heavy quarks on the lattice: status and perspectives

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1. – Introduction: Heavy quarks on the lattice

The violation of CP symmetry is one of the most important but still least understood features of the Standard Model. Much effort has been invested in order to analyse CP violation within the framework of the Cabibbo-Kobayashi-Maskawa (CKM) pattern of quark mixing. Heavy quark systems play an important rôle in the study of the CKM mixing matrix: they contain information on its least known elements and serve to test its parametrisation as a unitary 3×3 matrix in terms of three angles and one CP violating complex phase.

The phenomenological extraction of CKM matrix elements involving heavy quarks has been hampered by large hadronic uncertainties in the evaluation of current matrix elements appearing in the relevant weak decay amplitudes. Since quarks are confined within hadrons the exchange of soft gluons makes a perturbative analysis of weak decays impossible. During the past decade one has therefore sought to compute these matrix elements non-perturbatively in lattice QCD.

Lattice simulations of QCD are by now a mature field. They allow for the evaluation of hadronic masses, decay constants and form factors from first principles. However, before lattice results can be applied phenomenologically a critical assessment of systematic errors is required. In this lecture I shall present lattice results for leptonic and semi-leptonic decays of heavy quark systems as well as $B - \overline{B}$ mixing. Systematic effects and methods how to increase the accuracy of lattice data will be discussed. More detailed information can be found in recent review articles [1–4].

In order to formulate QCD on a discrete grid of points one approximates space-time by a euclidean, hypercubic lattice with lattice spacing a and volume $L^3 \cdot T$. One then

chooses a discretisation of the QCD action involving the quark fields $q(x), \overline{q}(x)$ and gauge fields $U_\mu(x) \in \mathrm{SU}(3), \mu = 1, \ldots, 4$. One such discretisation was formulated by Wilson [5] ("Wilson fermions") and has been used for almost all results I shall describe. Using the discretised QCD action S_{QCD}, one can define a partition function

$$\begin{aligned} Z &= \int D[U]D[\overline{q}]D[q]\, e^{-S_{\mathrm{QCD}}[U,\overline{q},q]} \\ &= \int D[U] \det Q\, e^{-S_G[U]}, \end{aligned} \tag{1}$$

where in the last line we have integrated out the quark fields, resulting in the determinant of the Wilson-Dirac operator times the exponentiated pure gauge action. The expectation value of an observable $\mathcal{O}$ is defined as

$$\langle \mathcal{O} \rangle = Z^{-1} \int D[U] \mathcal{O} \det Q\, e^{-S_G[U]}. \tag{2}$$

In a numerical simulation one evaluates $\langle \mathcal{O} \rangle$ by generating a representative sample of N_c gauge configurations in a Monte Carlo procedure. The expectation value $\langle \mathcal{O} \rangle$ is then approximated by the sample average $\overline{\mathcal{O}}$

$$\langle \mathcal{O} \rangle \simeq \overline{\mathcal{O}} = \frac{1}{N_c} \sum_{i=1}^{N_c} \mathcal{O}_i, \tag{3}$$

where $\mathcal{O}_i$ is the value of the observable computed on the ith configuration. Since one is working with a finite number of configurations, $\overline{\mathcal{O}}$ is obtained with a statistical error proportional to $1/\sqrt{N_c}$. This procedure yields the observable for non-zero values of the lattice spacing a. The continuum result is obtained in the limit $a \to 0$. In practice, this usually means that the simulation must be repeated for several values of a so that the results can be extrapolated to $a = 0$.

The appearance of the determinant in eq. (1) presents a major obstacle for numerical simulations since it is highly non-local and thus its evaluation is a huge computational overhead. Before the advent of efficient algorithms it had been suggested to disregard its effects completely by setting $\det Q = 1$. This defines the so-called *Quenched Approximation* [6], which in physical terms corresponds to neglecting quark loops in the determination of $\langle \mathcal{O} \rangle$. Although of unknown quality, the quenched approximation is still widely used and also accounts for most of the material covered here.

Since the (inverse) lattice spacing acts as an ultraviolet cutoff, its value in physical units places constraints on the scales that one is able to study. On present computers typical values lie in the range of $a^{-1} = 2 - 4\,\mathrm{GeV}$. These relatively low values of the cutoff imply that one expects large cutoff effects for the charm quark whose mass is not too far below a^{-1} [GeV]. More importantly, the b quark cannot be studied directly since its mass lies above the cutoff. The following methods are used used to circumvent this problem:

- Reduction of lattice artefacts

- Static approximation
- Non-relativistic QCD (NRQCD)

In the first approach one seeks to cancel the leading cutoff effects of order a in the Wilson action by employing so-called improved actions and operators [7–11]. For instance, the $\mathrm{O}(a)$ improved lattice action is defined as

$$S^{I}_{\mathrm{QCD}}[U,\overline{q},q] = S_{\mathrm{QCD}}[U,\overline{q},q] + c_{\mathrm{sw}}\frac{ia}{4}\sum_{x,\mu,\nu}\overline{q}(x)\sigma_{\mu\nu}F_{\mu\nu}(x)q(x), \tag{4}$$

where c_{sw} is an improvement coefficient and $F_{\mu\nu}$ is a lattice transcription of the field tensor. Further coefficients appear in the definitions of the improved axial and vector currents. Provided that all improvement coefficients are chosen appropriately one can show that lattice artefacts of $\mathrm{O}(a)$ are cancelled completely in masses and matrix elements. In a different approach, which can be applied for improved and unimproved actions, a reduction of lattice artefacts is achieved through absorbing higher-order effects of the quark mass into a rescaling factor [12]

$$q(x) \longrightarrow e^{am_{\mathrm{P}}/2}q(x), \qquad am_{\mathrm{P}} = \ln(1+am_{\mathrm{q}}), \tag{5}$$

where am_{q} is the bare quark mass. This normalisation is called the El-Khadra-Kronfeld-Mackenzie (EKM) norm. It must be emphasised that neither improvement nor the EKM norm solve the problem that the b quark cannot be studied directly. However, they reduce lattice artefacts around the charm quark mass, so that the obtained results can be extrapolated to the b quark mass much more reliably.

The static approximation is based on the leading term of the expansion of the heavy quark propagator in powers of the inverse heavy quark mass $1/m_Q$. Thus, one regards the b quark as infinitely heavy in this approach and expects that results will be subject to corrections of order $\Lambda_{\mathrm{QCD}}/m_Q$. Finally, NRQCD is an effective theory based on an expansion of the QCD action in the four-velocity of the heavy (non-relativistic) quark. Again one expects corrections to the effective theory whose influence on physics results has to be assessed.

From this discussion it is clear that none of the above methods gives an entirely satisfactory description of heavy quarks on the lattice. However, they all provide complementary information which can be used to reveal the full picture.

Besides lattice artefacts, another important source of systematic errors is the explicit breaking of chiral symmetry by the Wilson action. As a consequence, lattice versions of the local vector and axial currents are not conserved. Instead they are related to their continuum counterparts by finite renormalisations Z_{V} and Z_{A}, respectively. Although Z_{V} and Z_{A} have been computed non-perturbatively for $\mathrm{O}(a)$ improved actions [13, 14] these factors are known only in one-loop perturbation theory in the unimproved case. Furthermore, explicit chiral symmetry breaking causes operators with definite chirality – such as the four-fermion operator used to describe $B-\overline{B}$ mixing – to mix with operators of opposite chirality. Therefore, several matrix elements have usually to be determined

on the lattice and subsequently matched to the continuum matrix element. The general expression in the unimproved theory thus is

$$\langle f|\mathcal{O}|i\rangle^{\rm cont} = \sum_\alpha Z_\alpha \langle f|\mathcal{O}_\alpha^{\rm latt}|i\rangle + O(a), \tag{6}$$

where the Z_α's are the appropriate normalisation factors, and lattice artefacts of order a arise through mixing with higher dimension operators.

Finally, lattice estimates of dimensionful quantities are subject to uncertainties in the lattice scale. They are due to the fact that different quantities like $f_\pi, M_\rho, \ldots$, which are commonly used to set the scale a^{-1} in physical units give different results. This is closely related to using the quenched approximation, since loop effects are not expected to be the same for different quantities.

2. – Leptonic decays of heavy-light mesons

The leptonic decay constant f_P of a heavy-light pseudoscalar meson is related to the matrix elements of the axial current on the lattice via

$$\langle 0|A_4(0)|{\rm PS}\rangle = M_P f_P/Z_{\rm A}, \tag{7}$$

where M_P is the pseudoscalar mass and $Z_{\rm A}$ is the renormalisation factor of the lattice axial current. Both the matrix element and M_P are obtained from the asymptotic behaviour of the euclidean correlation function of the axial current at large separation t

$$\sum_{\vec{x}} \langle A_4(\vec{x},t) A_4^\dagger(0)\rangle \stackrel{t\gg 0}{\simeq} \frac{|\langle 0|A_4(0)|{\rm PS}\rangle|^2}{2M_P} \left\{ e^{-M_P t} + e^{-M_P(T-t)} \right\}. \tag{8}$$

The decay constant f_P can then be studied at several different values of the mass of the heavy quark. This enables one to study some predictions by the Heavy Quark Effective Theory (HQET). It is well known that HQET predicts the following scaling law in the limit of an infinitely heavy quark

$$f_P\sqrt{M_P} \stackrel{m_Q\to\infty}{\longrightarrow} {\rm const} \times \alpha_s(M_P)^{-2/\beta_0}, \tag{9}$$

where α_s is the strong coupling constant and $\beta_0 = 11 - 2n_f/3$. In order to test the quality of this prediction we plot in Fig. 1 the quantity

$$\Phi(M_P) = f_P\sqrt{M_P} \left(\frac{\alpha_s(M_P)}{\alpha_s(M_B)} \right)^{2/\beta_0} \tag{10}$$

as a function of $1/M_P$, using data in the static approximation [15–17] and for relativistic heavy quarks [16, 18, 19]. Using the static approximation as the limiting case, the figure illustrates that there are large corrections in $1/M_P$ to the scaling law, provided that lattice artefacts have been treated, either by using improvement (i.e. $c_{\rm sw} \geq 1$) or by employing the EKM norm. Failure to address the problem of lattice artefacts leads to an

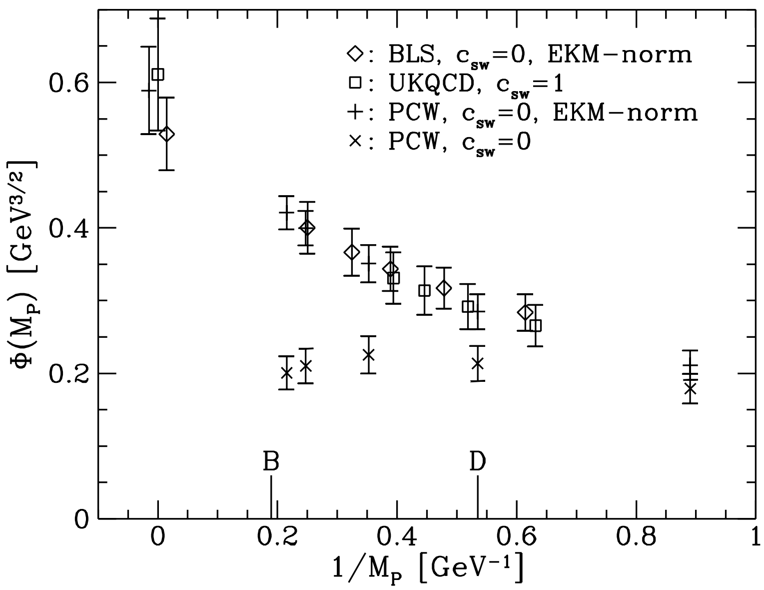

Fig. 1. – $\Phi(M_P)$ versus $1/M_P$ using data from different collaborations for $a^{-1} \simeq 3\,\mathrm{GeV}$. The data from [19] are shown with (plus signs) and without (crosses) including the EKM factors.

inconsistent mass behaviour of the data in the static and relativistic regimes. (c.f. crosses in Fig. 1). We conclude that the HQET scaling law is not satisfied at the physical B and D meson masses and that the treatment of lattice effects is crucial for the computation of heavy-light decay constants in general. Moreover, this example illustrates the interplay between different formulations of heavy quarks.

Recent results for heavy-light decay constants using relativistic heavy quarks and NRQCD are compiled in Table I. Besides the statistical error, most groups now quote one or more systematic errors. Note, however, that the estimation of systematic errors can vary significantly among different collaborations. The results shown in Table I are broadly consistent, but the errors are still large and dominated by systematic effects. Despite the apparent consistency of the data it would be premature to simply combine them into global estimates of decay constants, because the details of the analysis of the raw lattice data can differ substantially among different groups. A common analysis of lattice data from many collaborations has been described in [4]. There the aim was to perform the extrapolation to the continuum limit and to present a uniform estimation of systematic errors from a number of sources. These include the choice of lattice scale (e.g. $f_\pi, M_\rho, \ldots$), the quark field normalisation, uncertainties in the perturbative values of Z_{A}, and variations in fitting and extrapolation procedures. The results in the continuum limit can be summarised as

$$f_D = 191 \pm 19\,(\mathrm{stat})\,{}^{+\,3}_{-20}\,(\mathrm{syst})\,\mathrm{MeV}, \qquad f_{D_s} = 206 \pm 17\,(\mathrm{stat})\,{}^{+\,6}_{-22}\,(\mathrm{syst})\,\mathrm{MeV} \tag{11}$$

Collab.	a [fm]	f_D [MeV]	f_{D_s} [MeV]	f_B [MeV]	f_{B_s} [MeV]
FNAL* [20]	0	205(9)(27)	215(7)(30)	166(10)(28)	1.17(4)(3)
JLQCD* [21]	0	$192(10)\,^{+11}_{-16}$	$213(11)\,^{+12}_{-18}$	$163(12)\,^{+13}_{-16}$	
MILC* [22]	0	$186(10)\,^{+27}_{-18}\,^{+9}_{-0}$	$199(8)\,^{+40}_{-11}\,^{+10}_{-\ 0}$	$153(10)\,^{+36}_{-13}\,^{+13}_{-\ 0}$	$1.10(2)\,^{+5}_{-3}\,^{+3}_{-2}$
APE [23]	0.07	221(17)	237(16)	180(32)	1.14(8)
LANL [24]	0.09	$229(7)\,^{+20}_{-16}$	$260(4)\,^{+27}_{-22}$		
PCW [19]	0	170(30)		180(50)	1.09(2)(5)
UKQCD [18]	0.07	$185\,^{+4}_{-3}\,^{+42}_{-\ 7}$	$212(4)\,^{+46}_{-\ 7}$	$160(6)\,^{+59}_{-19}$	$1.22\,^{+4}_{-3}$
BLS [16]	0.06	208(9)(35)(12)	230(5)(10)(19)	187(10)(34)(15)	1.11(6)
Hirosh.† [25]	0.12			184(7)(5)(37)(37)	1.23(3)(3)
GLOK† [26]	0.09			183(32)(28)(16)	1.17(7)
SGO† [27]	0.10			126–166	1.24(4)(4)

TABLE I. – *Results for heavy-light decay constants from different collaborations. Data marked by an asterisk are preliminary. Results obtained using NRQCD are marked by a dagger. All other collaborations have used relativistic heavy quarks. The convention $f_\pi = 131\,\mathrm{MeV}$ is understood.*

$$f_B = 172 \pm 24\,(\mathrm{stat})\,^{+13}_{-19}\,(\mathrm{syst})\,\mathrm{MeV}, \qquad f_{B_s}/f_B = 1.14(8) \tag{12}$$

Further details and comparisons to other theoretical results can be found in [4].

3. – $B - \overline{B}$ mixing

We now turn our attention to matrix elements which are used to describe oscillations between B^0 and $\overline{B}^0$ states. The mechanism of $B^0 - \overline{B}^0$ mixing is illustrated by the box diagrams shown in Fig. 2.

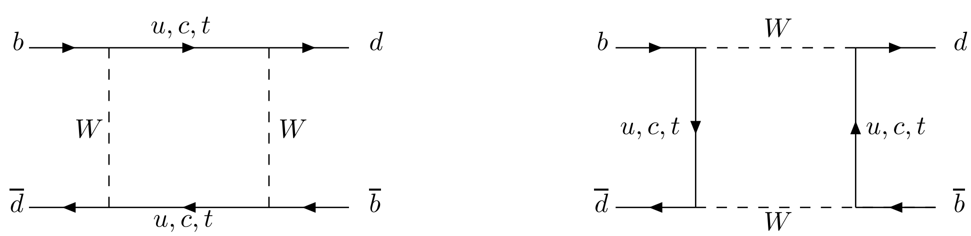

Fig. 2. – Box diagram contributions to $B^0 - \overline{B}^0$ mixing.

The mass difference Δm between the mass eigenstates B^0 and $\overline{B}^0$ is related to the CKM matrix elements V_{td} and V_{tb} via

$$\Delta m \propto |V_{td}V_{tb}^*|^2\, M_B f_B^2 \widehat{B}_B. \tag{13}$$

Here, f_B is the decay constant of the B meson encountered in the previous section, and

B_B denotes the B parameter defined by

$$B_B(\mu) = \frac{\langle \overline{B}^0 | \widehat{O}_L(\mu) | B^0 \rangle}{\frac{8}{3} f_B^2 M_B^2}, \tag{14}$$

where μ is the renormalisation scale and the four-fermion operator $\widehat{O}_L$ is given by

$$\widehat{O}_L = \left(\bar{b}\gamma_\mu(1-\gamma_5)d\right)\,\left(\bar{b}\gamma_\mu(1-\gamma_5)d\right). \tag{15}$$

The "hat" on B_B in eq. (13) signifies that the dependence of B_B on the renormalisation scale has been divided out. The resulting renormalisation group invariant B parameter can be defined at leading (LO) or next-to-leading order (NLO) in α_s via

$$\widehat{B}_B^{\mathrm{LO}} = \alpha_s(\mu)^{-2/\beta_0} B_B(\mu) \tag{16}$$

$$\widehat{B}_B^{\mathrm{NLO}} = \alpha_s(\mu)^{-2/\beta_0} \left(1 + \frac{\alpha_s(\mu)}{4\pi} J_5\right) B_B(\mu), \tag{17}$$

where J_5 is derived from the one- and two-loop anomalous dimensions of the operator $\widehat{O}_L$. It is important to realise that the combination $f_B^2 \widehat{B}_B$ is the principal unknown quantity which relates the experimentally measured mass difference Δm to the CKM matrix elements in eq. (13). Thus, $f_B^2 \widehat{B}_B$ is an important ingredient for the study of CP violation.

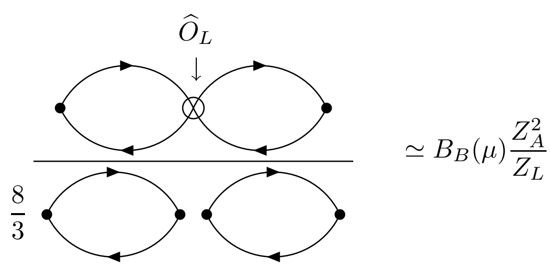

Fig. 3. – Lattice measurable for the contribution of $\widehat{O}_L$ to the B parameter (see text). Full dots represent insertions of the axial current.

On the lattice $B_B(\mu)$ is obtained from a ratio of two- and three-point functions as depicted in Fig. 3. Apart from the renormalisation constant Z_{A} of the lattice axial current the relation between the lattice and continuum versions of B_B will also contain the factor Z_L associated with the lattice version of $\widehat{O}_L$. Furthermore, as there is mixing between $\widehat{O}_L$ and other four-fermion operators with opposite and mixed chiralities, all contributions have to be evaluated and matched to the continuum theory. Since the relevant Z-factors have only been computed in one-loop perturbation theory, the matching procedure introduces considerable uncertainties into the final estimates of the B parameter. For lattice results obtained in the static approximation and using $c_{\mathrm{sw}} = 1$, these uncertainties have been estimated to be as large as 25% [4, 17, 28].

Collab.	$a\,[\mathrm{fm}]$	$B_B(m_b)$	$\widehat{B}_B^{\mathrm{LO}}$	$\widehat{B}_B^{\mathrm{NLO}}$
Kentucky† [30]	0.09	0.97(4)	1.42(6)	1.54(6)
G+M† [28]	0.09	0.63(4)	0.92(6)	1.00(6)
	0.09	0.73(4)	1.07(6)	1.16(6)
UKQCD† [17]	0.07	$0.69\,^{+3}_{-4}\,^{+2}_{-1}$	$1.02\,^{+5}_{-6}\,^{+3}_{-2}$	$1.10\,^{+5}_{-6}\,^{+3}_{-2}$
UKQCD† [29]	0.07	$0.83\,^{+3}_{-4}\,^{+2}_{-1}$		$1.32\,^{+5}_{-7}\,^{+3}_{-2}$
JLQCD* [31]	0.06	0.840(60)	1.23(9)	1.34(10)
	0.08	0.895(47)	1.31(7)	1.42(8)
B+S* [32]	0	0.89(6)(4)	1.30(9)(6)	1.42(10)(6)
ELC [33]	0.05	0.84(5)	1.24(7)	1.34(8)
BDHS [34]	0.08	0.93(14)	1.36(20)	1.48(22)

TABLE II. – *Data for the B parameter from different collaborations at a reference scale $m_b = 5\,\mathrm{GeV}$. Data marked by a dagger have been obtained using the static approximation. Data marked by an asterisk are preliminary.*

Lattice data for $B_B(\mu)$ have been published for propagating and static heavy quarks and are shown in Table II. It is clear that lattice data for B_B do not yet allow for a continuum extrapolation as in the case of f_B. Instead one may quote a common estimate with an error that encompasses the spread of different results. In ref. [4] the result is

$$B_B(5\,\mathrm{GeV}) = 0.85\,^{+13}_{-22}, \qquad \widehat{B}_B = 1.3\,^{+2}_{-3}. \tag{18}$$

The global result for the ratio B_{B_s}/B_B quoted in [4] is

$$B_{B_s}/B_B = 1.00 \pm 0.02. \tag{19}$$

We can now combine the results for the decay constant f_B in eq. (12) with that for the B parameter. Combining the errors in quadrature one finds

$$f_B\sqrt{\widehat{B}_B} = 195\,^{+30}_{-40}\,\mathrm{MeV}. \tag{20}$$

For the SU(3)-flavour breaking ratio involving both the decay constants and B parameters the global result in [4] is

$$\frac{f_{B_s}^2 B_{B_s}}{f_B^2 B_B} = 1.38 \pm 0.15. \tag{21}$$

These estimates can now be used in the study of the CKM matrix and CP violation.

4. – Semi-leptonic $B \to \pi$ and $B \to \rho$ decays

We will now discuss lattice results for semi-leptonic decays of B mesons. Due to lack of space we will concentrate on heavy-to-light transitions. Semi-leptonic $B \to D$ and $B \to D^*$ decays are reviewed, for instance, in ref. [1].

Decays like $\overline{B}^0 \to \pi^+\ell^-\overline{\nu}_\ell$ or $\overline{B}^0 \to \rho^+\ell^-\overline{\nu}_\ell$ have attracted much interest recently, since they can be used to extract V_{ub}, which is one of the most poorly known CKM matrix elements. Compared to the pseudoscalar decay constant or the B parameter, the study of semi-leptonic decays is more complicated due to the kinematics involved in their description. The relevant matrix elements of the weak $V - A$ current are parametrised in terms of form factors which depend on the momentum transfer q^2 between the initial and final mesons. For instance, if the final state is a pseudoscalar meson only the vector current contributes, and there are two independent form factors f^+ and f^0

$$\langle \mathrm{PS}(k)\,|V_\mu|\,B(p)\rangle = f^+(q^2)\left\{(p+k)_\mu - \frac{M_B^2 - M_P^2}{q^2}q_\mu\right\} + f^0(q^2)\frac{M_B^2 - M_P^2}{q^2}q_\mu, \tag{22}$$

where $q_\mu = p_\mu - k_\mu$. If the final state is a vector meson, there are four form factors $V(q^2), A_1(q^2), A_2(q^2)$ and $A_0(q^2)$, associated with matrix elements of the vector and axial vector currents, respectively.

The need to control lattice artefacts places constraints on the possible values of lattice momenta $\vec{p}$ and $\vec{k}$. Together with the constraints on the heavy quark mass this implies that one usually obtains form factors for typical momenta $|\vec{p}| \leq 1.5\,\mathrm{GeV}/c$ and heavy quark masses in the region of charm. The "generic" heavy-to-light semi-leptonic decay in lattice simulations is thus $D \to K\ell\nu_\ell$ for momentum transfers in the range $-1\,\mathrm{GeV}^2/c^2 \leq q^2 \leq 2\,\mathrm{GeV}^2/c^2$. Although in principle one is interested in the q^2-dependence of form factors, they are commonly quoted at $q^2 = 0$. Frequently a simple pole ansatz is used to model the q^2-dependence:

$$F(q^2) = \frac{F(0)}{(1 - q^2/M_{\mathrm{pole}}^2)^{n_F}}. \tag{23}$$

Here, F is a generic form factor, M_{pole} is of the order of the heavy-light meson mass, and $n_F = 0, 1, 2, \ldots$ parametrises constant, monopole, dipole and higher multipole behaviour of F. A recent compilation of lattice results for form factors for semi-leptonic D decays can be found in [1,35]. Since the lattice form factors for these decays are determined in a region around $q^2 = 0$, the pole ansatz in eq. (23) merely serves to guide the interpolation of $F(q^2)$ to $q^2 = 0$, so that no model dependence is introduced.

The situation changes significantly if one considers semi-leptonic $B \to \pi$ or $B \to \rho$ decays. Here the form factors are obtained through extrapolation in the heavy quark mass to the mass of the b quark. Since similarly large values of lattice momentum $|\vec{p}|$ cannot be considered due to restrictions imposed by lattice artefacts, the accessible region of q^2 is pushed to large values near q^2_{max}. This in turn leaves a long and potentially uncontrollable extrapolation in q^2 to determine $F(0)$. In order to map out the q^2-behaviour one usually cannot avoid relying on model assumptions.

We shall now describe how the q^2-behaviour can be constrained by requiring consistency with heavy quark symmtery (HQS), kinematical constraints and scaling laws implied by light-cone sum rules. In analogy to the decay constant in eq. (9), HQS predicts the following leading scaling behaviour of form factors in the infinite mass limit at

Collab.	a [fm]	$f^+(0)$	$V(0)$	$A_1(0)$	$A_2(0)$
UKQCD [38]	0.07	0.27(11)	$0.35\,^{+6}_{-5}$	$0.27\,^{+5}_{-4}$	$0.25\,^{+5}_{-3}$
GSS [39]	0.06	0.43(19)	0.65(15)	0.28(3)	0.46(23)
APE [40]	0.09	0.35(8)	0.53(31)	0.24(12)	0.27(80)
ELC [41]	0.05	0.30(14)(5)	0.37(11)	0.22(5)	0.49(21)(5)

Table III. – *Lattice results for form factors for semi-leptonic $B \to \pi$ and $B \to \rho$ decays.*

fixed values of ω (which is the product of four-velocities of the initial and final mesons):

$$f^+(\omega) \sim M^{1/2}, \quad f^0(\omega) \sim M^{-1/2}, \quad V(\omega) \sim M^{1/2}, \quad A_1(\omega) \sim M^{-1/2}, \ldots \tag{24}$$

Here, M is the mass of the heavy-light meson. The extrapolations of form factors to the b quark mass can be performed using model functions motivated by the above scaling laws. Additional scaling laws are provided by light-cone sum rule analyses [36,37], which predict that all form factors scale like $F \sim M^{-3/2}$ at $q^2 = 0$. In the heavy quark limit

$$1 - \frac{q^2}{M_{\rm pole}^2} \sim \frac{1}{M}, \tag{25}$$

and thus the scaling laws predicted by HQS and light-cone sum rules can be combined to infer a value for n_F in the pole formula eq. (23). Also, kinematical constraints at $q^2 = 0$ such as

$$f^+(0) = f^0(0) \tag{26}$$

can be used to analyse the q^2-dependence of lattice form factors. In Table III we list lattice estimates for form factors for $\overline{B}^0 \to \pi^+\ell^-\overline{\nu}_\ell$ and $\overline{B}^0 \to \rho^+\ell^-\overline{\nu}_\ell$ decays. It should be noted that only the UKQCD data are consistent with all constraints discussed above.

A very different approach to constrain the q^2-dependence in a model-independent fashion has been discussed by Lellouch [42]. Lattice data for the form factors f^+ and f^0 for $\overline{B}^0 \to \pi^+\ell^-\overline{\nu}_\ell$ obtained near $q^2_{\rm max}$ have been combined with dispersion relations and the kinematical constraint eq. (26). The method relies on perturbative QCD in the evaluation of the dispersion relations and general properties such as unitarity, analyticity and crossing. However, existing lattice data for the form factors are at present not precise enough in order to allow for stringent bounds at $q^2 = 0$.

Another proposal to avoid model dependence in the extraction of V_{ub} was made in [43]. Here one concentrates on the *exclusive* decay $\overline{B}^0 \to \rho^+\ell^-\overline{\nu}_\ell$ in the region near $q^2_{\rm max}$. Instead of attempting to extract the form factors at $q^2 = 0$ one parametrises the differential decay rate by

$$\frac{d\Gamma}{dq^2} = 10^{-12}\frac{G_F^2|V_{ub}|^2}{192\pi^3 M_B^3} q^2\sqrt{\lambda(q^2)}\mathcal{A}^2\big(1 + \mathcal{B}\big(q^2 - q^2_{\rm max}\big)\big), \tag{27}$$

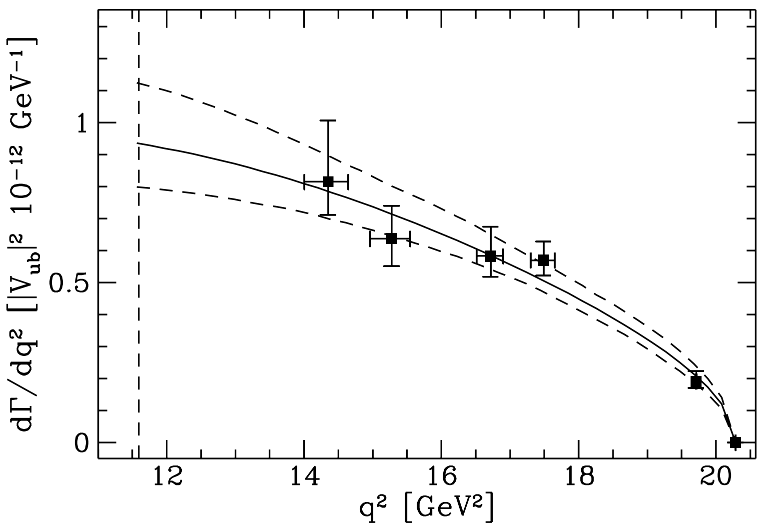

Fig. 4. – Differential decay rate for $\overline{B}^0 \to \rho^+ \ell \overline{\nu}_\ell$ from [43]. Points are lattice data, and the fit and variation in eq. (28) is represented by the solid and dashed curves, respectively. The vertical dashed line marks the endpoint of charm production.

where $\mathcal{A}$ and $\mathcal{B}$ are parameters and λ is a phase-space factor. The combination $\mathcal{A}^2(1 + \mathcal{B}\left(q^2 - q^2_{\rm max}\right))$ parametrises the long-distance hadronic dynamics, and $\mathcal{A}^2$ provides the overall normalisation. Using lattice data for the form factors to evaluate the differential decay rate the authors of [43] find

$$\mathcal{A} = 4.6\,^{+0.4}_{-0.3} \pm 0.6\,\mathrm{GeV}, \qquad \mathcal{B} = (-8\,^{+4}_{-6}) \cdot 10^{-2}\,\mathrm{GeV}^2. \tag{28}$$

The corresponding prediction of the decay rate is shown in Fig. 4. Given sufficient experimental data for the decay rate in conjunction with accurate lattice results, a determination of V_{ub} will be possible.

* * *

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